Computation of the induced norm from L to L∞ in SISO sampled-data systems: Discretization approach with convergence rate analysis

Kim, Jung Hoon; Hagiwara, Tomomichi

Computation of the Induced Norm from L_2 to L_∞ in SISO Sampled-Data Systems: Discretization Approach with Convergence Rate Analysis

Jung Hoon Kim and Tomomichi Hagiwara

***Abstract*— This paper provides a discretization method for computing the induced norm from L_2 to L_∞ in single-input/single-output (SISO) linear time-invariant (LTI) sampled-data systems. We first follow the lifting-based treatment for the induced norm from L_2 to L_∞ of SISO LTI sampled-data systems, but further apply the key idea of fast-lifting, by which the sampling interval $[0, h)$ is divided into M subintervals with an equal width. Such an idea allows us to develop two methods for computing the induced norm with gridding and piecewise constant approximations. These methods leads to approximately equivalent discretization methods of the generalized plant that can be used for readily computing upper and lower bounds of the induced norm together with the derivation of the associated convergence rates. More precisely, it is shown that the approximation error converges to 0 at the rate of $1/\sqrt{M}$ and $1/M$ in the gridding and piecewise constant approximation methods, respectively.**

I. Introduction

Sampled-data systems [1]–[19] taking into account of their intersample behavior occur naturally in feedback control applications. A number of studies on sampled-data systems associated with the disturbance rejection problem have been conducted [1]–[19]. Many of these studies deal with the performance analysis for disturbances in L_2 and consider the L_2 norm of the output since such analysis corresponds to the H_∞ analysis. However, computing the L_∞ norm (i.e., the maximum amplitude) of the output instead could equally play an important role and some studies [4]–[6] indeed deal with the induced norm from L_2 to L_∞. This induced norm admits an equivalent alternative interpretation as the H_2 norm in the single-input/single-output (SISO) linear time-invariant (LTI) systems, both for continuous-time and discrete-time cases, [20]–[23], even though the H_2 norm (of a multi-input/multi-output LTI system) is usually defined in the frequency domain and related to the power of the output for a white noise input. In this connection, it is worth noting that, for sampled-data systems, two definitions of their H_2 norm have been given in [7]–[10] and each of them gives a different generalization of the definition of the H_2 norm of continuous-time systems.

The induced norm from L_2 to L_∞ in LTI sampled-data systems (consisting of LTI plants and LTI controllers) was analytically formulated first in [4] by using the lifting technique [1]–[4], but no explicit computation method for the induced norm was given in that study. An explicit computation method for the induced norm without the lifting treatment was provided in [5], but its comparison with the two definitions for the H_2 norm of sampled-data systems was not discussed there. Recently, the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems was revisited in [6] again with the lifting arguments in such a way that the comparison of the induced norm with the two existing definitions for the H_2 norm of sampled-data systems is easy. The arguments therein showed that the induced norm coincides with neither of the two existing definitions of the H_2 norm for sampled-data systems, and thus the induced norm from L_2 to L_∞ could be interpreted as yet another definition of the H_2 norm of LTI sampled-data systems. However, to compute the induced norm with the arguments in [6], one should take the supremum of a suitably constructed function over the sampling interval $[0, h)$, for which only an approximate computation approach based on gridding is given and the same kind of situation applies also to the arguments in [5].

This paper extends the arguments in [6] by developing two methods for computing upper bound (as well as lower bound) of the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems through the ideas of gridding and piecewise constant approximations under the fast-lifting treatment [13], in which the sampling interval $[0, h)$ is divided into M subintervals with an equal width. More precisely, it is shown for each approximation method that upper and lower bounds of the induced norm from L_2 to L_∞ in a SISO LTI sampled-data system can be obtained through computing the induced norm form l_2 to l_∞ of an approximately equivalent LTI discrete-time system constructed with the associated approximate discretization of the continuous-time plant. The latter discrete-time induced norm is readily computable [23], and it is further shown that the gaps between the upper and lower bounds converge to 0 at the rates of $1/\sqrt{M}$ and $1/M$ in the gridding and piecewise constant approximations, respectively, where M is the fast-lifting parameter.

The notations $\mathbb{N}$, $\mathbb{R}^\nu_\infty$ and $\mathbb{R}^\nu_2$ are used to denote the set of positive integers, the Banach space of ν-dimensional real vectors equipped with vector ∞-norm and the Hilbert space of ν-dimensional real vectors equipped with the usual inner product and the associated Euclidean norm, respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N} \cup \{0\}$. The notation

J. H. Kim is with the Center for Robotics Research, Korea Institute of Science and Technology (KIST), 5, Hwarango-ro 14-gil, Seongbuk-gu, Seoul 136-791, Republic of Korea (e-mail:kimjunghoon28@gmail.com).

T. Hagiwara is with the Department of Electrical Engineering, Kyoto University, Nishikyo-ku, Kyoto 615-8510, Japan (e-mail:hagiwara@kuee.kyoto-u.ac.jp).

$\|\cdot\|_\infty$ is used to mean either the $L_\infty[0,h)$ norm of a vector function, i.e., $\|f(\cdot)\|_\infty := \max_j \operatorname*{ess\,sup}_{0\le t<h} |f_j(t)|$ or that with h replaced by an integer fraction $h' = h/M$, whose distinction will be clear from the context. On the other hand, the notation $\|\cdot\|_2$ is used to mean either the $L_2[0,h)$ norm of a vector function, i.e., $\|f(\cdot)\|_2 := \left(\int_0^h f^T(t)f(t)dt\right)^{1/2}$, (or that with h replaced by $h' = h/M$ or ∞), the induced norm from either l_2 or $L_2[0,h')$ to $\mathbb{R}_2^\nu$, the Euclidean norm of a vector, or the induced (i.e., 2) norm of a matrix as a mapping from $\mathbb{R}_2^{\nu_1}$ to $\mathbb{R}_2^{\nu_2}$, whose distinction will be clear from the context. The notation $\|\cdot\|_{\infty/2}$ is used to mean, unless stated otherwise, the induced norm from either $L_2[0,h')$ or $\mathbb{R}_2^\nu$ to $L_\infty[0,h')$ or the induced norm of a matrix as a mapping from $\mathbb{R}_2^{\nu_1}$ to $\mathbb{R}_\infty^{\nu_2}$, whose distinction will be clear from the context. Furthermore, we call the induced norms from L_2 to L_∞ and from l_2 to l_∞ the L_∞/L_2-induced norm and l_∞/l_2-induced norm, respectively, for simplicity. The notation $\|\cdot\|_{\infty/2}$ is used also for these induced norms. Finally, the notations $\|\cdot\|_F$ and $\|\cdot\|_{H_2}$ denote the Frobenious norm of a matrix and H_2 norm of a discrete-time system, respectively.

II. Characterization of the L_∞/L_2-Induced Norm in SISO LTI Sampled-Data Systems

Let us consider the stable sampled-data system Σ_{SD} shown in Fig. 1, where P denotes the continuous-time LTI generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time and signals discrete-time signals, respectively. Suppose that P and Ψ are given respectively by

$$P: \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \tag{1}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}_2^n$, $w(t) \in \mathbb{R}_2$, $u(t) \in \mathbb{R}_2^{n_u}$, $z(t) \in \mathbb{R}_\infty$, $y(t) \in \mathbb{R}_2^{n_y}$, $\psi_k \in \mathbb{R}_2^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$.

We apply the lifting technique [1]–[4] to the sampled-data system Σ_{SD}; given $f \in L_\infty$ or $f \in L_2$, its lifting $\{\widehat{f}_k\}_{k=0}^\infty$ with $\widehat{f}_k \in L_\infty[0,h)$ or $L_2[0,h)$ $(k \in \mathbb{N}_0)$ is defined by

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \le \theta < h) \tag{3}$$

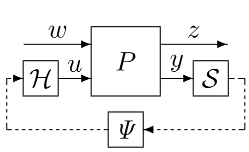

Fig. 1. Sampled-data system Σ_{SD}.

By applying lifting to $w \in L_2$ and $z \in L_\infty$, the lifted representation of the sampled-data system Σ_{SD} is given by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{4}$$

with $\xi_k := [x_k^T \ \psi_k^T]^T$ $(x_k := x(kh))$, the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}_2^{n+n_\Psi} \to \mathbb{R}_2^{n+n_\Psi} \tag{5}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : L_2[0,h) \to \mathbb{R}_2^{n+n_\Psi} \tag{6}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}_2^{n+n_\Psi} \to L_\infty[0,h) \tag{7}$$

$$\mathcal{D} = \mathbf{D}_{11} : L_2[0,h) \to L_\infty[0,h) \tag{8}$$

where

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta,\ C_{2d} := C_2 \tag{9}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} : \mathbb{R}_2^n \to \mathbb{R}_2^{n+n_\Psi},\ C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{10}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta)d\theta \tag{11}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = C_0 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix} \tag{12}$$

$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix},\ C_0 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \tag{13}$$

$$(\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau)d\tau \tag{14}$$

We first note (4) and describe the closed-loop relation between $\widehat{w}_k$ and $\widehat{z}_k$ $(k = 0, \cdots, \infty)$ as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & \\ \mathcal{C}\mathcal{B} & \mathcal{D} & 0 & \cdots & \\ \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{B} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \vdots \end{bmatrix} \tag{15}$$

The L_∞/L_2-induced norm of the sampled-data system Σ_{SD} coincides with the associated induced norm of the above operator in the right hand side. In particular, since this operator has a Toeplitz structure, it readily follows from $\|z\|_\infty = \sup_k \|\widehat{z}_k\|_\infty$ that the L_∞/L_2-induced norm of Σ_{SD} coincides with the L_∞/L_2-induced norm of its "last" block row, i.e., (after reordering without affecting the induced norm)

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{C}\mathcal{B} & \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{A}^2\mathcal{B} & \cdots \end{bmatrix} \tag{16}$$

Here, with a slight abuse of the terminology, the L_∞/L_2-induced norm of $\mathcal{F}$ refers to

$$\begin{aligned} \|\mathcal{F}\|_{\infty/2} &:= \sup_{\|\widehat{w}\|_2 \le 1} \|(\mathcal{F}\widehat{w})(\cdot)\|_\infty \\ &= \sup_{\|\widehat{w}\|_2 \le 1} \sup_{0\le\theta<h} |(\mathcal{F}\widehat{w})(\theta)| = \sup_{0\le\theta<h} \sup_{\|\widehat{w}\|_2 \le 1} |(\mathcal{F}\widehat{w})(\theta)| \end{aligned} \tag{17}$$

where $\widehat{w} := [\widehat{w}_0,\ \widehat{w}_1,\ \cdots]^T$, and $\|\widehat{w}\|_2$ denotes $(\sum_{k=0}^\infty \|\widehat{w}_k\|^2)^{1/2}$ (which actually equals the L_2 norm

$\|w\|_2$). To explicitly characterize the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$, we briefly sketch the results in [6] as follows. For each $\theta \in [0,h]$, we first introduce the matrices

$$W_\theta := \int_0^\theta \exp(A(\theta-\tau))B_1B_1^T \exp(A^T(\theta-\tau))d\tau \quad (18)$$

$$C_\theta := C_0 \exp(A_2\theta)C_\Sigma \quad (19)$$

Then, with W_h, consider the solution X_h to the discrete-time Lyapunov equation

$$\mathcal{A}X_h\mathcal{A}^T - X_h + \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad (20)$$

Then, it was shown that

$$\sup_{\|\widehat{w}\|_2 \le 1} |(\mathcal{F}\widehat{w})(\theta)| = F(\theta) := (C_1W_\theta C_1^T + C_\theta X_h C_\theta^T)^{1/2} \quad (21)$$

This together with (17) leads to the following result.

Theorem 1 ([6]): The L_∞/L_2-induced norm associated with the SISO LTI sampled-data system $\Sigma_{\rm SD}$ is given by

$$\|\mathcal{F}\|_{\infty/2} = \sup_{0\le\theta<h} F(\theta) \quad (22)$$

Even though Theorem 1 gives an almost direct computation method for the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$, this theorem cannot lead to an easily obtainable upper bound of the induced norm $\|\mathcal{F}\|_{\infty/2}$. Furthermore, when a lower bound of $\|\mathcal{F}\|_{\infty/2}$ is obtained through a gridding idea, it is not clear how much the lower bound could deviate from the exact norm in the worst case. In this regard, the following section is devoted to giving a readily computable upper bound as well as a lower bound of the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$.

III. Explicit Computation of $\|\mathcal{F}\|_{\infty/2}$

This section aims at computing upper and lower bounds of $\|\mathcal{F}\|_{\infty/2}$ by using gridding approximation and piecewise constant approximation approaches. In particular, both approaches are interpreted and developed through the fast-lifting treatment [13] of signals on the interval $[0,h)$. Here, with $M \in \mathbb{N}$ (the fast-lifting parameter) and $h' := h/M$, fast-lifting is defined as the mapping from $f \in L_\infty[0,h)$ or $f \in L_2[0,h)$ to $\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(M)})^T]$ belonging to $(L_\infty[0,h'))^M$ or $(L_2[0,h'))^M$, and is denoted by $\check{f} = \mathbf{L}_M f$, where

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h') \quad (23)$$

We note that $\mathbf{L}_M$ is norm-preserving both on $L_\infty[0,h)$ and $L_2[0,h)$, from which it readily follows that

$$\|\mathcal{F}\|_{\infty/2} = \|\mathcal{F}_M\|_{\infty/2} \quad (24)$$

where the right hand side denotes the associated induced norm of

$$\mathcal{F}_M := \begin{bmatrix} \mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1} & \mathbf{L}_M\mathcal{C}\mathcal{B}\mathbf{L}_M^{-1} & \mathbf{L}_M\mathcal{C}\mathcal{A}\mathcal{B}\mathbf{L}_M^{-1} & \cdots \end{bmatrix} \quad (25)$$

With a slight abuse of terminology again, we call $\|\mathcal{F}_M\|_{\infty/2}$ the L_∞/L_2 induced norm of $\mathcal{F}_M$, for simplicity. To facilitate the treatment of the right-hand side of (25), we introduce $\mathbf{D}'_{11}$, $\mathbf{B}'_1$ and $\mathbf{M}'_1$ defined as $\mathbf{D}_{11}$, $\mathbf{B}_1$ and $\mathbf{M}_1$, respectively, with the horizon $[0,h)$ replaced by $[0,h')$, and also introduce the matrices

$$A'_d := \exp(Ah'), \ \ A'_{2d} := \exp(A_2h')$$

$$J := \begin{bmatrix} I \\ 0 \end{bmatrix} : \mathbb{R}_2^n \to \mathbb{R}_2^{n+n_u} \quad (26)$$

Then, $\mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1}$ and $\mathbf{L}_M\mathcal{C}\mathcal{A}^j\mathcal{B}\mathbf{L}_M^{-1}$ $(j=0,1,\cdots)$ in (25) are given respectively by

$$\mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1}\Delta_M^0\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \quad (27)$$

$$\mathbf{L}_M\mathcal{C}\mathcal{A}^j\mathcal{B}\mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1}A'_{2dM}C_\Sigma\mathcal{A}^jJ_\Sigma A'_{dM}\overline{\mathbf{B}'_1} \quad (28)$$

where

$$A'_{dM} := \begin{bmatrix} (A'_d)^{M-1} & \cdots & I \end{bmatrix}, \ A'_{2dM} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix}$$

$$\Delta_M^0 := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{2d})^{M-2}J & \cdots & J & 0 \end{bmatrix} \quad (29)$$

and $\overline{(\cdot)}$ denotes ${\rm diag}[(\cdot),\cdots,(\cdot)]$ consisting of M copies of $(\cdot)$. Hence, the operator matrix on the right hand side of (25) admits the representation

$$\mathcal{F}_M = \begin{bmatrix} \overline{\mathbf{M}'_1}\Delta_M^0\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} & \overline{\mathbf{M}'_1}\mathcal{J}_{M0}\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_1}\mathcal{J}_{M1}\overline{\mathbf{B}'_1} & \cdots \end{bmatrix} \quad (30)$$

where $\mathcal{J}_{Mj} := A'_{2dM}C_\Sigma\mathcal{A}^jJ_\Sigma A'_{dM} \quad (j=0,1,\cdots)$.

We are in a position to develop two methods for computing upper and lower bounds of the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2} = \|\mathcal{F}_M\|_{\infty/2}$ by approximately dealing with the operators $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$.

A. Gridding Approximation Approach

We first introduce the operator $\mathbf{M}'_{\rm a0} : \mathbb{R}_2^{n+n_u} \to \mathbf{L}_\infty[0,h')$ given by

$$\left(\mathbf{M}'_{\rm a0}\begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = C_0\begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le \theta' < h') \quad (31)$$

The output of $\mathbf{M}'_{\rm a0}$ is a constant function corresponding to the zero-order approximation of the Taylor expansion of the output of $\mathbf{M}'_1$, and this operator is used also in the piecewise constant approximation arguments in the following subsection. In the present subsection, however, we rather note an alternative simple interpretation that its output is obtained by holding the output of $\mathbf{M}'_1$ at $\theta' = 0$. We then consider the operator $\mathcal{F}_{M\rm G}$ obtained by replacing $\mathbf{M}'_1$ with $\mathbf{M}'_{\rm a0}$ and ignoring $\mathbf{D}'_{11}$ in (30). In other words, we define

$$\mathcal{F}_{M\rm G} = \begin{bmatrix} \overline{\mathbf{M}'_{\rm a0}}\Delta_M^0\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_{\rm a0}}\mathcal{J}_{M0}\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_{\rm a0}}\mathcal{J}_{M1}\overline{\mathbf{B}'_1} & \cdots \end{bmatrix} \quad (32)$$

By noting that the output of $\mathbf{D}'_{11}$ equals 0 at $\theta' = 0$, it readily follows from the above interpretation of $\mathbf{M}'_{\rm a0}$ that $\mathcal{F}_{M\rm G}$ shares the same fast-lifted output as $\mathcal{F}_M$ for the same input as long as we evaluate both outputs at $\theta' = 0$. By noting the arguments behind Theorem 1, we readily see that

$$\|\mathcal{F}_{MG}\|_{\infty/2} = \max_{\theta=0,h',\cdots,(M-1)h'} F(\theta) \tag{33}$$

In other words, computing $\|\mathcal{F}_{MG}\|_{\infty/2}$ corresponds to a very simple gridding treatment of the function $F(\theta)$ with M equally spaced points. Hence, in the following, we call the operator $\mathcal{F}_{MG}$ the gridding approximation of $\mathcal{F}$.

By (22) and (33), it is obvious that $\|\mathcal{F}_{MG}\|_{\infty/2}$ gives a lower bound approximation of $\|\mathcal{F}\|_{\infty/2}$ for each $M \in \mathbb{N}$. Furthermore, an exact computation of $\|\mathcal{F}_{MG}\|_{\infty/2}$ is straightforward if we again note its alternative characterization (33), together with (21). Indeed, this approximate computation method of $\|\mathcal{F}\|_{\infty/2}$ is nothing but the one given in [6], but this method is not necessarily satisfactory in the following two respects. First, it does not provide a method for giving an upper bound of $\|\mathcal{F}\|_{\infty/2}$. Secondly, the computation formula has no clear relationship with an associated (discrete-time) dynamical system, so that it is not clear how we could extend the computation method to arguments for designing controllers minimizing the L_∞/L_2-induced norm. Hence, the remainder of this subsection is devoted to dealing directly with $\mathcal{F}_{MG}$ (rather than the alternative characterization (33)) so that the above two issues can be resolved, and the following three lemmas are important for such directions.

Lemma 1: For $\mathbf{D}'_{11} : L_2[0,h/M) \to L_\infty[0,h/M)$, its induced norm is given by $\|\mathbf{D}'_{11}\|_{\infty/2} = (C_1 W_{h'} C_1^T)^{1/2}$ and admits the representation

$$\|\mathbf{D}'_{11}\|_{\infty/2} = \frac{K_{MD}}{\sqrt{M}} \tag{34}$$

where K_{MD} has a uniform upper bound with respect to M given by $K_D^U := \sqrt{h}\|C_1\|_{\infty/2} e^{\|A\|_2 h}\|B_1\|_2$.

Lemma 2: The inequality

$$\|\overline{(\mathbf{M}'_1 - \mathbf{M}'_{a0})}\Delta_M^0 \overline{\mathbf{B}'_1}\|_{\infty/2} \le \frac{K_{MD0}}{M} \tag{35}$$

holds with

$$K_{MD0} := h\|C_1 A\|_{\infty/2} e^{\|A\|_2 h/M}\|V_{(M-1)h'}\|_2 \tag{36}$$

where the matrix V_θ is defined by

$$V_\theta V_\theta^T = W_\theta \quad (\theta \in [0,h]) \tag{37}$$

Furthermore, K_{MD0} has a uniform upper bound with respect to M given by $K_{D0}^U := h\|C_1 A\|_{\infty/2} e^{\|A\|_2 h}\|V_h\|_2$.

Lemma 3: The inequality

$$\|\overline{(\mathbf{M}'_1 - \mathbf{M}'_{a0})} A'_{2dM}\|_{\infty/2} \le \frac{K_{MC0}}{M} \tag{38}$$

holds, where

$$K_{MC0} := h\|C_0 A_2\|_{\infty/2} e^{\|A_2\|_2 h/M} \cdot \max_{1\le i\le M} \|(A'_{2d})^{i-1}\|_2 \tag{39}$$

Furthermore, K_{MC0} has a uniform upper bound with respect to M given by $K_{C0}^U := h\|C_0 A_2\|_{\infty/2} e^{\|A_2\|_2 h}$.

From Lemmas 1–3, we readily have the following result.

Theorem 2: Let us consider the discrete-time system

$$\Sigma_{\mathrm{D}} : \begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + J_\Sigma V_h w_k \\ z_k &= C_\Sigma \xi_k \end{cases} \tag{40}$$

with the matrix V_h defined by (37), and denote the H_2 norm of the discrete-time system Σ_{D} by $\|\Sigma_{\mathrm{D}}\|_{H_2}$. Then, the inequality

$$\|\mathcal{F}_M - \mathcal{F}_{MG}\|_{\infty/2} \le \frac{K_{M0}}{M} + \frac{K_{MD}}{\sqrt{M}} \tag{41}$$

holds, where

$$K_{M0} := K_{MD0} + K_{MC0} \cdot \|\Sigma_{\mathrm{D}}\|_{H_2} \tag{42}$$

Furthermore, K_{M0} has a uniform upper bound with respect to M given by $K_0^U := K_{D0}^U + K_{C0}^U \cdot \|\Sigma_{\mathrm{D}}\|_{H_2}$, while K_{MD} has a uniform upper bound with respect to M given by K_D^U.

Since $\|\mathcal{F}_{MG}\|_{\infty/2}$ can be computed exactly (recall (33)), Theorem 2 clearly gives an upper bound of $\|\mathcal{F}\|_{\infty/2} = \|\mathcal{F}_M\|_{\infty/2}$ through a triangle inequality, where the gap between such an upper bound and the aforementioned lower bound $\|\mathcal{F}_{MG}\|_{\infty/2}$ obviously converges to 0 at the rate of $1/\sqrt{M}$ as $M \to \infty$ in this gridding approximation method.

We next give an alternative method for exactly computing $\|\mathcal{F}_{MG}\|_{\infty/2}$ through discretization treatment of the generalized plant P. By (31), the output of $\mathcal{F}_{MG}$ is a constant function determined by the matrix C_0. This immediately implies that the induced norm $\|\mathcal{F}_{MG}\|_{\infty/2}$ coincides with the induced norm of the infinite-dimensional matrix

$$\begin{aligned} &[\overline{C_0}\Delta_M^0 \overline{V_{h'}} \quad \overline{C_0}\mathcal{J}_{M0}\overline{V_{h'}} \quad \overline{C_0}\mathcal{J}_{M1}\overline{V_{h'}} \quad \cdots] \\ &= [\overline{C_0}\Delta_M^0 \overline{V_{h'}} \quad C_{M0} J_\Sigma A'_{dM}\overline{V_{h'}} \quad C_{M0}\mathcal{A} J_\Sigma A'_{dM}\overline{V_{h'}} \quad \cdots] \end{aligned} \tag{43}$$

from l_2 to $\mathbb{R}_\infty^M$, where

$$C_{M0} = \overline{C_0} A'_{2dM} C_\Sigma : \mathbb{R}_2^{n+n_\Psi} \to \mathbb{R}_\infty^M \tag{44}$$

Here, let us consider the unit-ball image of $\mathbf{B}_1 : L_2[0,h) \to \mathbb{R}_2^n$. Then, on one hand, it coincides with the unit-ball image of $V_h : \mathbb{R}_2^{n_V} \to \mathbb{R}_2^n$ (where n_V is the number of columns of $V_{h'}$) while (by the norm-preserving property of $\mathbf{L}_M$) it also coincides with the unit-ball image of $\mathbf{B}_1\mathbf{L}_M^{-1} = A'_{dM}\overline{\mathbf{B}'_1}$, or equivalently, that of $A'_{dM}\overline{V_{h'}} : \mathbb{R}_2^{Mn_V} \to \mathbb{R}_2^n$. Hence, the unit-ball image of $A'_{dM}\overline{V_{h'}}$ coincides with that of V_h and thus the induced norm of (43) coincides with that of

$$[\overline{C_0}\Delta_M^0 \overline{V_{h'}} \quad C_{M0} J_\Sigma V_h \quad C_{M0}\mathcal{A} J_\Sigma V_h \quad \cdots] \tag{45}$$

from l_2 to $\mathbb{R}_\infty^M$. Note from the structure of Δ_M^0 that $\overline{C_0}\Delta_M^0 = \overline{C_1}\Delta_{M,n}^0$ holds, where $\Delta_{M,n}^0$ is defined as Δ_M^0 with J and A'_{2d} replaced by I and A'_d, respectively. Noting that the induced norm from l_2 to $\mathbb{R}_\infty^M$ may be computed through the row-wise independent treatment of (45), we can apply essentially the same arguments as those used in deriving (45) from (43). More precisely, we see from the structure of $\Delta_{M,n}^0$ that the induced norm of (45) equals that of

$$F_{MG} := [D_{MG} \quad C_{M0} J_\Sigma V_h \quad C_{M0}\mathcal{A} J_\Sigma V_h \quad \cdots] \tag{46}$$

from l_2 to $\mathbb{R}^M_\infty$ by defining

$$D_{MG} = \overline{C_1}\begin{bmatrix}0 & V_{h'}^T & V_{2h'}^T & \cdots & V_{(M-1)h'}^T\end{bmatrix}^T : \mathbb{R}_2^{n_V} \to \mathbb{R}^M_\infty \tag{47}$$

Summarizing all the above arguments, we arrive at the conclusion that $\|\mathcal{F}_{MG}\|_{\infty/2}$ equals the induced norm of F_{MG} from l_2 to $\mathbb{R}^M_\infty$ in (46). This gives an alternative exact computation method of $\|\mathcal{F}_{MG}\|_{\infty/2}$ and further leads to the following observation. Namely, because the matrix F_{MG} corresponds to the 'last' block row of the infinite-dimensional Toeplitz matrix representation of the input/output relation of the discrete-time system

$$\Sigma_{MG}: \begin{cases}\xi_{k+1} &= \mathcal{A}\xi_k + J_\Sigma V_h w_k\\ z_k &= C_{M0}\xi_k + D_{MG}w_k\end{cases} \tag{48}$$

we readily have the following result giving an alternative characterization of the gridding approximation $\|\mathcal{F}_{MG}\|_{\infty/2}$.

Theorem 3: The gridding approximation $\|\mathcal{F}_{MG}\|_{\infty/2}$ of the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ coincides with the l_∞/l_2-induced norm of the discrete-time system Σ_{MG}, i.e., $\|\mathcal{F}_{MG}\|_{\infty/2} = \|\Sigma_{MG}\|_{\infty/2}$.

Corollary 1: The following inequality holds.

$$\|\Sigma_{MG}\|_{\infty/2} \le \|\mathcal{F}\|_{\infty/2} \le \|\Sigma_{MG}\|_{\infty/2} + \frac{K_{M0}}{M} + \frac{K_{MD}}{\sqrt{M}} \tag{49}$$

We note that the l_∞/l_2-induced norm can be easily computed [23]. Furthermore, we readily see that the discrete-time system Σ_{MG} coincides with the closed-loop system obtained by connecting Ψ to the discrete-time plant

$$P_{MG}: \begin{cases}x_{k+1} &= A_d x_k + V_h w_k + B_{2d}u_k\\ z_k &= C_{Md}x_k + D_{MG}w_k + D_{Md}u_k\\ y_k &= C_{2d}x_k\end{cases} \tag{50}$$

where the matrices $C_{Md}: \mathbb{R}_2^n \to \mathbb{R}^M_\infty$ and $D_{Md}: \mathbb{R}_2^{n_u} \to \mathbb{R}^M_\infty$ are given by

$$\begin{bmatrix}C_{Md} & D_{Md}\end{bmatrix} := C_{M0}A'_{2dM} \tag{51}$$

Hence, the above results are important in showing that the controller synthesis problem for minimizing the L_∞/L_2-induced norm of sampled-data systems is approximately reducible to the discrete-time l_∞/l_2-induced norm problem through the discretized generalized plant P_{MG} in (50).

B. Piecewise Constant Approximation Approach

We next consider the piecewise constant approximation $\mathcal{F}_{M0}$ of $\mathcal{F}$. This is defined as

$$\mathcal{F}_{M0} = \begin{bmatrix}\overline{\mathbf{M}'_{a0}}\Delta_M^0\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} & \overline{\mathbf{M}'_{a0}}\mathcal{J}_{M0}\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_{a0}}\mathcal{J}_{M1}\overline{\mathbf{B}'_1} & \cdots\end{bmatrix} \tag{52}$$

and the only difference from the gridding approximation in the preceding subsection is that $\overline{\mathbf{D}'_{11}}$ in $\mathcal{F}_M$ in (30) is retained as it is. It turns out that this simple difference leads to an improved convergence rate with respect to the fast-lifting parameter M. From Lemmas 2 and 3, we have the following result.

Theorem 4: The following inequality holds.

$$\|\mathcal{F}_M - \mathcal{F}_{M0}\|_{\infty/2} \le \frac{K_{M0}}{M} \tag{53}$$

Furthermore, K_{M0} has a uniform upper bound with respect to M given by K_0^U.

This theorem clearly implies that the approximation error in the piecewise constant approximation approach to the computation of $\|\mathcal{F}\|_{\infty/2} = \|\mathcal{F}_M\|_{\infty/2}$ by $\|\mathcal{F}_{M0}\|_{\infty/2}$ converges to 0 at the rate of $1/M$ as $M \to \infty$. In this context, it is most important that we can have an exact computation method for $\|\mathcal{F}_{M0}\|_{\infty/2}$, as discussed in the following.

We first note that Δ_M^0 is strictly block lower triangular and thus so is $\overline{\mathbf{M}'_{a0}}\Delta_M^0\overline{\mathbf{B}'_1}$, while $\overline{\mathbf{D}'_{11}}$ is block diagonal. Furthermore, it is obvious from the definition of the $L_\infty[0, h')$ norm that $\|\mathcal{F}_{M0}\|_{\infty/2}$ equals $\max_i \|\mathcal{F}_{M0i}\|_{\infty/2}$, where $\mathcal{F}_{M0i}$ $(i = 1, \cdots, M)$ are the ith block row of $\mathcal{F}_{M0}$. From these facts, it is not hard to see that $\mathcal{F}_{M0}$ may be redefined as

$$\mathcal{F}_{M0} = \begin{bmatrix}\overline{\mathbf{D}'_{11}} & \overline{\mathbf{M}'_{a0}}\Delta_M^0\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_{a0}}\mathcal{J}_{M0}\overline{\mathbf{B}'_1} & \overline{\mathbf{M}'_{a0}}\mathcal{J}_{M1}\overline{\mathbf{B}'_1} & \cdots\end{bmatrix} \tag{54}$$

without changing the induced norm. Note that the output of each entry of $\mathcal{F}_{M0}$ in (54) is a constant function by the definition of $\overline{\mathbf{M}'_{a0}}$ except that of $\overline{\mathbf{D}'_{11}}$, where $\mathbf{D}'_{11}$ is simply a convolution integral operator on $[0, h')$. On the other hand, as an induced norm of time-invariant operators, the $L_\infty[0,\tau)/L_2[0,\tau)$-induced norm of $\mathbf{D}_{11}$ in (14) restricted to the time interval $[0, \tau)$ is nondecreasing with respect to τ. Combining these observations (in particular, since the entries in (54) other than $\mathbf{D}'_{11}$ have constant outputs), it is not hard to see that the induced norm $\|\mathcal{F}_{M0}\|_{\infty/2}$ corresponds to the output of $\mathcal{F}_{M0}$ in (54) for $\theta' \to h'$ for the worst input (even though it is not the case, in general, for the original $\|\mathcal{F}_M\|_{\infty/2}$). Hence, we are led to consider the output of $\mathbf{D}'_{11}$ only for $\theta' \to h'$, and it obviously coincides with the output of $C_1\mathbf{B}'_1$ for the same input. This implies that $\overline{\mathbf{D}'_{11}}$ in (54) may be replaced by $\overline{C_1\mathbf{B}'_1}$ without affecting the induced norm. Hence, by repeating the arguments about the unit-ball images of $\mathbf{B}'_1$ and $V_{h'}$, we see that the induced norm of (54) coincides with that of

$$\begin{bmatrix}\overline{C_1V_{h'}} & D_{MG} & C_{M0}J_\Sigma V_h & C_{M0}\mathcal{A}J_\Sigma V_h & \cdots\end{bmatrix} \tag{55}$$

from l_2 to R^M_∞. Note that the only difference from F_{MG} in (46) is the existence of the first entry $\overline{C_1V_{h'}}$, and thus the rest of the arguments is essentially the same as those in the gridding approximation; the only issues we should take care is that this entry has M times as many columns as others and also corresponds to increasing the number of row of the direct feedthrough matrix $[\overline{C_1V_{h'}}\ D_{MG}]$ (compared with D_{MG} in the gridding approximation approach). To circumvent these issues at once, it follows from the property of $\mathbb{R}^M_\infty$ that considering instead the induced norm of

$$F_{M0} := \begin{bmatrix}D_{M0} & C_{M0}J_\Sigma\widetilde{V}_h & C_{M0}\mathcal{A}J_\Sigma\widetilde{V}_h & \cdots\end{bmatrix} \tag{56}$$

from l_2 to $\mathbb{R}^M_\infty$ suffices with

$$D_{M0} := \begin{bmatrix} \widetilde{D}_0 & D_{M\mathrm{G}} \end{bmatrix} : \mathbb{R}_2^{2n_V} \to \mathbb{R}^M_\infty \tag{57}$$

$$\widetilde{V}_h := \begin{bmatrix} 0 & V_h \end{bmatrix} : \mathbb{R}_2^{2n_V} \to \mathbb{R}_2^n \tag{58}$$

where the matrix $\widetilde{D}_0$ is described as

$$\widetilde{D}_0 = \begin{bmatrix} (C_1 V_{h'})^T & \cdots & (C_1 V_{h'})^T \end{bmatrix}^T : \mathbb{R}_2^{n_V} \to \mathbb{R}^M_\infty \tag{59}$$

To conclude, $\|\mathcal{F}_{M0}\|_{\infty/2}$ equals the induced norm of F_{M0} from l_2 to $\mathbb{R}^M_\infty$ in (56). Since each entry therein has the same number of columns, essentially the same arguments as in the gridding approximation lead to the assertion that $\|\mathcal{F}_{M0}\|_{\infty/2}$ equals the l_∞/l_2-induced norm of the discrete-time system

$$\Sigma_{M0} : \begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + J_\Sigma \widetilde{V}_h w_k \\ z_k &= C_{M0}\xi_k + D_{M0} w_k \end{cases} \tag{60}$$

That is, we readily have the following result giving an alternative characterization of the piecewise constant approximation.

Theorem 5: The piecewise constant approximation $\|\mathcal{F}_{M0}\|_{\infty/2}$ of the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ coincides with the l_∞/l_2-induced norm of the discrete-time Σ_{M0}, i.e., $\|\mathcal{F}_{M0}\|_{\infty/2} = \|\Sigma_{M0}\|_{\infty/2}$.

Corollary 2: The following inequality holds.

$$\|\Sigma_{M0}\|_{\infty/2} - \frac{K_{M0}}{M} \le \|\mathcal{F}\|_{\infty/2} \le \|\Sigma_{M0}\|_{\infty/2} + \frac{K_{M0}}{M} \tag{61}$$

Here, we note that the discrete-time system Σ_{M0} coincides with the closed-loop system obtained by connecting Ψ to the discrete-time plant

$$P_{M0} : \begin{cases} x_{k+1} &= A_d x_k + \widetilde{V}_h w_k + B_{2d} u_k \\ z_k &= C_{Md} x_k + D_{M0} w_k + D_{Md} u_k \\ y_k &= C_{2d} x_k \end{cases} \tag{62}$$

Hence, the controller design problem for minimizing the L_∞/L_2-induced norm of the sampled-data system Σ_{SD} is reducible to the discrete-time l_∞/l_2-induced norm problem through the above discretized plant P_{M0}.

IV. Conclusion

This paper provided two methods for computing the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems through gridding and piecewise constant approximations, which are stimulated by the success in our preceding study [6] relevant to explicitly characterizing and computing a lower bound of the induced norm. These methods allow us to compute an upper bound as well as a lower bound of the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems through approximately equivalent discretization of the plant (and the corresponding l_∞/l_2-induced norm computation available in existing studies) and thus can be extended to the associated controller synthesis problem (as a future topic). It was shown that the gaps between the bounds converge to 0 at the rate of $1/\sqrt{M}$ and $1/M$ in the gridding and piecewise constant approximations, respectively, where M is the fast-lifting parameter with which the sampling interval is fractioned in the analysis.

References

[1] Y. Yamamoto, A function space approach to sampled-data control systems and their tracking problems, *IEEE Trans. Automat. Contr.*, Vol. 39, No. 4, pp. 703–713, 1994.

[2] B. Bamieh and J. B. Pearson, A general framework for linear periodic systems with application to H^∞ sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. 37, No. 4, pp. 418–435, 1992.

[3] H. T. Toivonen, Sampled-data control of continuous-time systems with an H_∞ optimality criterion, *Automatica*, Vol. 28, No. 1, pp. 45–54, 1992.

[4] B. Bamieh, J. B. Pearson, B. A. Francis and A. Tannenbaum, A lifting technique for linear periodic systems with applications to sampled-data systems, *Syst. Control Lett.*, Vol. 17, No. 2, pp. 79–88, 1991.

[5] G. G. Zhu and R. E. Skelton, L_2 to L_∞ gains for sampled-data systems, *Int. J. Control*, Vol. 61, No. 1, pp. 19–32, 1995.

[6] J. H. Kim and T. Hagiwara, Induced norm from L_2 to L_∞ in SISO sampled-data systems, *Proc 2015 American Control Conference*, pp. 2862–2867, 2015.

[7] T. Chen and B. A. Francis, H_2-optimal sampled-data control, *IEEE Trans. Automat. Contr.*, Vol. 36, No. 4, pp. 387–397, 1991.

[8] B. Bamieh and J. B. Pearson, The H^2 problem for sampled-data systems, *Syst. Control Lett.*, Vol. 19, No. 1, pp. 1–12, 1992.

[9] P. P. Khargonekar and N. Sivashankar, H_2 optimal control for sampled-data systems, *Syst. Control Lett.*, Vol. 17, No. 6, pp. 425–436, 1991.

[10] T. Hagiwara and M. Araki, FR-operator approach to the H_2 analysis and synthesis of sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. 40. No. 8, pp. 1411–1421, 1995.

[11] G. Tadmor, H_∞ optimal sampled-data control in continuous time systems, *Int. J. Control*, Vol. 56, No. 1, pp. 99–141, 1992.

[12] G. E. Dullerud, Computing the L_2-induced norm of a compression operator, *Syst. Control Lett.*, Vol. 7, No. 2, pp. 87–91, 1999.

[13] T. Hagiwara and H. Umeda, Modified fast-sample/fast-hold approximation for sampled-data system analysis, *Eur. J. Control*, Vol. 14, No. 4, pp. 286–296, 2008.

[14] G. E. Dullerud and B. A. Francis, L_1 analysis and design of sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. 37, No. 4, pp. 436–446, 1992.

[15] B. Bamieh, M. A. Dahleh and J. B. Pearson, Minimization of the L^∞-induced norm for sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. 38, No. 5, pp. 717–732, 1993.

[16] N. Sivashankar and P. P. Khargonekar, Induced norms for sampled-data systems, *Automatica*, Vol. 28, No. 6, pp. 1267–1272, 1992.

[17] J. H. Kim and T. Hagiwara, Computing the $L_\infty[0,h)$-induced norm of a compression operator, *Syst. Control Lett.*, Vol. 67, No. 1, pp. 1–8, 2014.

[18] J. H. Kim and T. Hagiwara, L_∞-induced norm analysis of sampled-data systems via piecewise constant and linear approximations, *Automatica*, Vol. 51, No. 1, pp. 223–232, 2015.

[19] J. H. Kim and T. Hagiwara, L_1 discretization for sampled-data controller synthesis via piecewise linear approximation, *IEEE Trans. Automat. Contr.*, accepted for publication.

[20] D. A. Wilson, Convolution and Hankel operator norms for linear systems, *IEEE Trans, Automat. Contr.*, Vol. 34, No. 1, pp. 94–97, 1989.

[21] M. A. Rotea, The generalized H_2 control problem, *Automatica*, Vol. 29, No. 2, pp. 373–385, 1993.

[22] V. Chellabonia, W. M. Haddad, D. S. Bernstein and D. A. Wilson, Induced convolution operator norms of linear dynamical systems, *Math. Control Signals Systems*, Vol. 13, No. 3, pp. 216–239, 2000.

[23] D. A. Wilson, M. A. Nekoui and G. D. Halikias, An LQR weight selection approach to the discrete generalized H_2 control problem, *Int. J. Control*, Vol. 71, No. 1, pp. 93–101, 1998.